Project of Research

Jonathan Chappelon

The Steinhaus graph $G(S)$, with n vertices, associated with the sequence of 0's and 1's $S = (a_1, \ldots, a_{n-1})$, of length $n-1$, is the simple graph whose adjacency matrix is the symmetric $(0,1)$-matrix $(a_{i,j})_{1 \leqslant i,j \leqslant n}$, with a diagonal of zeros, and whose upper triangular part is defined by $a_{1,j} = a_{j-1}$ for all j, $2 \leqslant j \leqslant n$, and by the relationship $a_{i,j} \equiv a_{i-1,j-1} + a_{i-1,j} \pmod 2$, for i and j such that $2 \leqslant i < j \leqslant n$. A general problem on Steinhaus graphs is that of characterizing those, or their associated binary sequence, satisfying a given graph property. The bipartite Steinhaus graphs were characterized in [2] and the planar ones in [7]. In 1979 [6], W. Dymacek conjectured that the only regular Steinhaus graphs are those generated by the sequences (1) (the complete graph with two vertices K_2), $(0, \ldots, 0)$ (the graph with n vertices and no edges) and the periodic sequences $(1, 1, 0, \ldots, 1, 1, 0)$, of period 110, and of lengths divisible by 3. This conjecture was verified up to 117 vertices in 2008 [1]. I managed, during my Ph.D. thesis, to extend the verification that K_2 is the only regular Steinhaus graph of odd degree up to 1500 vertices. The main idea is that the anti-diagonal entries of a Steinhaus matrix are determined by the vertex degrees of its associated Steinhaus graph. This result was published in [5]. The first goal of my project of research is to obtain a complete proof of this conjecture, at least for the regular Steinhaus graphs of odd degree.

The binary Steinhaus triangle ∇S associated with the sequence S is the upper triangular part of the adjacency matrix of the Steinhaus graph $G(S)$. This object can also be defined in any finite cyclic group $\mathbb{Z}/n\mathbb{Z}$. The Steinhaus triangle ∇S associated with a finite sequence S of length m in $\mathbb{Z}/n\mathbb{Z}$ appeared then as the collection of the m iterated derived sequences $\nabla S = \{S, \partial S, \partial^2 S, \ldots, \partial^{m-1} S\}$, where the derived sequence ∂S of S is the sequence of length $m-1$ in $\mathbb{Z}/n\mathbb{Z}$ obtained by pairwise adding consecutive terms of S. During my thesis, I worked to prove the existence, in any $\mathbb{Z}/n\mathbb{Z}$ for n odd, of balanced Steinhaus triangles, that are Steinhaus triangles in which all the elements of $\mathbb{Z}/n\mathbb{Z}$ appear with the same multiplicity. This problem was posed, for the first time, by H. Steinhaus in 1963 for $n = 2$ and was generalized by Molluzzo in 1976 [8]. It remained completely open for all $n \geqslant 3$ up to 2008. A complete and positive answer to this problem is given in [4] for all $n = 3^k$, with $k \geqslant 1$. I have obtained these solutions by analysing Steinhaus triangles associated with arithmetic sequences of $\mathbb{Z}/n\mathbb{Z}$. A Steinhaus triangle can also be seen as a figure appearing in the orbit $(\partial^i S)_{i \in \mathbb{N}}$ of a doubly infinite sequence S of $\mathbb{Z}/n\mathbb{Z}$, i.e. as a finite subset of a cellular automaton associated with the derivation function ∂ defined by $\partial S = (a_j + a_{j+1})_{j \in \mathbb{Z}}$ for every sequence $S = (a_j)_{j \in \mathbb{Z}}$ of $\mathbb{Z}/n\mathbb{Z}$. We consider Steinhaus figures as Steinhaus triangles, generalized Pascal triangles, lozenges, trapezoids or parallelograms. Determining the existence of balanced Steinhaus figures is then a natural problem. I managed to prove the existence of a sequence of integers, that I called universal sequence, whose projection orbit in $\mathbb{Z}/n\mathbb{Z}$, for n odd, contains infinitely many balanced Steinhaus figures of each kind. More precisely, I have explicitly built balanced Steinhaus triangles and balanced Pascal triangles for 2/3 of admissible sizes in every $\mathbb{Z}/n\mathbb{Z}$ for $n = p^k$, with p an odd prime and $k \geqslant 1$. These results appear in the article [3] which has been accepted for publication in Journal of Combinatorial Theory series A. The second goal of my project of research is to extend these results obtained on the balanced Steinhaus figures appearing in a cellular automaton defined from ∂, to Steinhaus figures appearing across linear cellular automata of dimension 1, that are those with a linear derivation function. This kind of function can be defined as follows: for every positive integer r and for all $(2r+1)$-tuple of integers $W = (w_{-r}, \ldots, w_0, \ldots, w_r)$, the derivation function ∂_W associated with a doubly infinite sequence $(a_j)_{j\in\mathbb{Z}}$ of $\mathbb{Z}/n\mathbb{Z}$ is the sequence

$$\partial_W (a_j)_{j\in\mathbb{Z}} = \left(\sum_{k=-r}^{r} w_k a_{j+k} \right)_{j\in\mathbb{Z}} .$$

The third and last goal of my project of research is to study the Steinhaus figures appearing in the cellular automaton of dimension 2 in which the standard Pascal tetrahedron appears. In this kind of automaton, we can define Steinhaus tetrahedra and generalized Pascal tetrahedra.

With the experience I have obtained during my three years of Ph.D. and my two years of postdoc, I think I can make substantial progress on these three goals in a period of 2 years.

References

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