RESEARCH STATEMENT

RATNIK GANDHI

My research interest lie in Algebra, Algorithms and Game Theory. My work is related to the study of rich and powerful structure available in the polynomial algebra, group theory and algebraic geometry, and developing algebraic methods for solving computational problems. My current research involves deriving alternate algebraic algorithms for the problems in Game theory.

COMPUTATION OF NASH EQUILIBRIA

Game theory is the study of rational decision-making in a competitive situation [MvN44], i.e. players choose their strategies to maximize their respective payoffs. A Nash equilibrium [Nas51] of the game is an outcome in which none of the players have any unilateral incentive to change their strategies. Every finite game has a mixed strategy Nash equilibrium [Nas51]. The construction or computation of a Nash equilibrium is an open problem in general.

In recent years, the problem of computing Nash equilibria has gained prominence, and has generated substantial research literature. The problem of computing a Nash equilibrium is shown to be PPAD-complete [CD06, DGP06]. The result suggests that the problem is likely to be hard in general, and it has given a major thrust to approximation methods [Rou10]. However new results suggest that there is no general polynomial time approximation scheme for computing a Nash equilibrium [DGP09].

PRESENT AND FUTURE WORK

In the view of the unavailability of an efficient method for computing Nash equilibria, my primary motivation is to study the structural aspects of Nash equilibria and use this knowledge to suggest new methods. The specific question that I work on is: Whether a method can be found for computing all equilibria of the input game, given a single equilibrium (sample equilibrium), without repeating the solution procedure for the sample equilibrium. For addressing this question we consider two subclasses of games. The class of *rational payoff irrational equilibria*(RPIE) games are games with all payoff values given by rational numbers while all equilibria are irrational numbers. The class of *integer payoff irrational equilibria*(IPIE) games are similarly defined. An example of the classes of games was first introduced in [SN50] and later in [NCH03]. The classes of games are important as they allow establishing relations among the equilibria of a game with the knowledge of Galois groups.

Based on characterization in [Stu02], I model the Nash equilibrium solutions of a game as solutions to a system of polynomial equations and call it the *game system* ($\mathcal{GS}$). I present algorithms for deciding membership to the classes of games and develop algorithms for computing their Nash equilibria using a sample equilibrium [CG10b, CG10a]. The algorithms are deterministic and give exact solutions.

Two Player Games

A substantial amount of games in practice can be modelled as two player games. The class of two player games have received special attention as they can be solved with system of linear equations over arbitrary fields [LH64]. However, no result of linearity of the games is known when defined over ring of integers $\mathbb{Z}$, or arbitrary rings. With enormous amount of existing research literature on two player games, the problem of generalizing results in [LH64] over rings becomes important and interesting. In my work, I show that no two player games can be a member of the class of RPIE games, and requires special treatment [CG10a].

Polynomial Time Membership Decision

The algorithm for deciding membership to the classes of games uses Gröbner basis [CG10b]. This increases computational complexity of the algorithm. For making the algorithm efficient I formulate a restriction for the polynomials in the ideal $\mathcal{I}$ of the $\mathcal{GS}$. The $\mathcal{GS}$ follows the restriction when its ideal $\mathcal{I}$ is radical. My experiments suggest that $\mathcal{I}$ for the RPIE and IPIE games are radical. If this conjecture of the radical property of the ideal is true, then the membership decision can be taken in polynomial time. This is significant improvement over the earlier approach. I propose to study this problem further as it has consequences on the equilibria computation and the game construction problems.

Equilibria in Closed Form

Given a real number, its storage in computer memory with high precision is an important problem. The reverse problem of deciding whether the stored finite precision number is a rational or irrational is also fundamental. The problem of storage of irrational equilibria of the class of RPIE games was studied in [LM04]. I show that for a subclass of RPIE games its Nash equilibria can be computed in algebraic(closed) form [CG10a]. The result follows from the radical field extension of field of rational number $\mathbb{Q}$ and Galois correspondence. I would like to investigate further the question: Are ring extensions of the ring of integers $\mathbb{Z}$, that I consider, radical? i.e., does the ring extension S over the ring of integers $\mathbb{Z}$ have following the form?

$$R = \mathbb{Z} = L_0 \subset L_1 \subset \ldots \subset L_n = S,$$

and $\exists\ \alpha_i \in L_{i+1}$, a natural number n_i, such that $L_{i+1} = L_i(a_i)$ and $\alpha_i^{n_i} \in L_i$. An affirmative answer to this question leads to a subclass of IPIE games whose equilibria can be solved in closed from.

The generalization of Galois theory over rings, given in [CHR65], allowed me to treat class of IPIE games separately. I derived the required machinery for the class of IPIE games. I prove that due to trivial Galois group of the rational number extension of the ring of integers, the algorithms for computing all equilibria can not be used for games with integer payoffs and rational equilibria. Further, lack of total order over finite fields, the algebra derived for classes of RPIE and IPIE games can not be extended to work over finite fields.

Construction of The Classes of Games

With derived structural results on the relations between equilibria solutions, we are better equipped to deal with more examples of the classes of games. I propose to work on the problem of construction of games considering the following approaches:

Explicit Construction. Given a set of solutions, the problem of computing a multivariate polynomial system with the set as its solutions is not new. The problem of constructing more examples of the classes of games can be considered analogous.i.e., Given an equilibrium set and with known structure of the $\mathcal{GS}$, how to compute the coefficients of the $\mathcal{GS}$ so that the $\mathcal{GS}$ represents a game with given set as its equilibria.

In my experiments, the approach raises the issue of non-equilibria solutions of the $\mathcal{GS}$. At present it is not known how to control the non-equilibria solutions that cause the constructed game to have more equilibria solutions than the given set. This is a polynomial algebra problem and is closely linked with the problem of constructing vanishing ideal for a given set of variety points.

Composition. Composition of games can be another approach. Consider any non-member game to be played in parallel with a known IPIE or RPIE game. The new game can be considered as a composition of the component games – a non-member game and a known IPIE or RPIE game. It would be interesting to investigate the restrictions on the payoff values of the component games that guarantee the composite game to be a member of either class of games.

Perturbation. Given an input known IPIE or RPIE game, we perturb its payoff values by independent and identically distributed random variables. With what probability we are guaranteed to get the perturbed game a member to the either class of games. If perturbation ρ guarantees perturbed game to be a member of the classes of games with some positive probability p, then it is possible to analyse smoothed complexity [CDT06] of the algorithms that I present.

Extensive and Graphical Form Games

The results I presented are addressed towards a subclass of normal form games. They are in the form of interrelations between the Nash equilibria solutions of a given game. These relations link all the equilibria solutions through solutions of the $\mathcal{GS}$. I propose to further investigate extensions of these results so that all the Nash equilibria are linked directly. An approach for the investigation would be to define set of automorphisms, similar to Galois groups, for the multivariate polynomial systems of the form $\mathcal{GS}$.

Further, I would like to investigate necessary modifications to my results so that they can be extended to work with a larger class of normal form, extensive form and graphical form games.

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